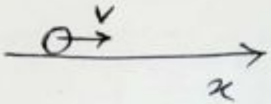
Probabilistic Graphical Models

Lectures 10

How to represent CPDs, Continusous Case

Continuous Variables

Constant velocity



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$$x_{t+1} = x_t + \Delta t \, v \quad \Rightarrow \quad \begin{aligned} x_{t+1} &= x_t + \Delta_t V_t \\ V_{t+1} &= V_t \end{aligned}$$

deterministic

$$\Downarrow$$

$$x_{t+1} \simeq x_t + \Delta_t V_t$$

$$V_{t+1} \simeq V_t$$

non-deterministic

How to fomalize?

Additive Noise Model

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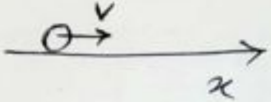
Constant velocity



$$x_{t+1} = x_t + \Delta t \, v \quad \Rightarrow \quad x_{t+1} = x_t + \Delta_t V_t$$

$$V_{t+1} = V_t$$

deterministic

$$\Downarrow$$

$$x_{t+1} \simeq x_t + \Delta_t V_t$$

$$V_{t+1} \simeq V_t$$

non-deterministic

How to fomalize?

Additive Noise Model

$X \rightarrow Y$

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X, Y $\quad Y = f(X)$ deterministic

$Y \simeq f(X)$ non-deterministic

A common solution is a deterministic formula + additive noise

$Y = f(X) + n \rightarrow$ noise

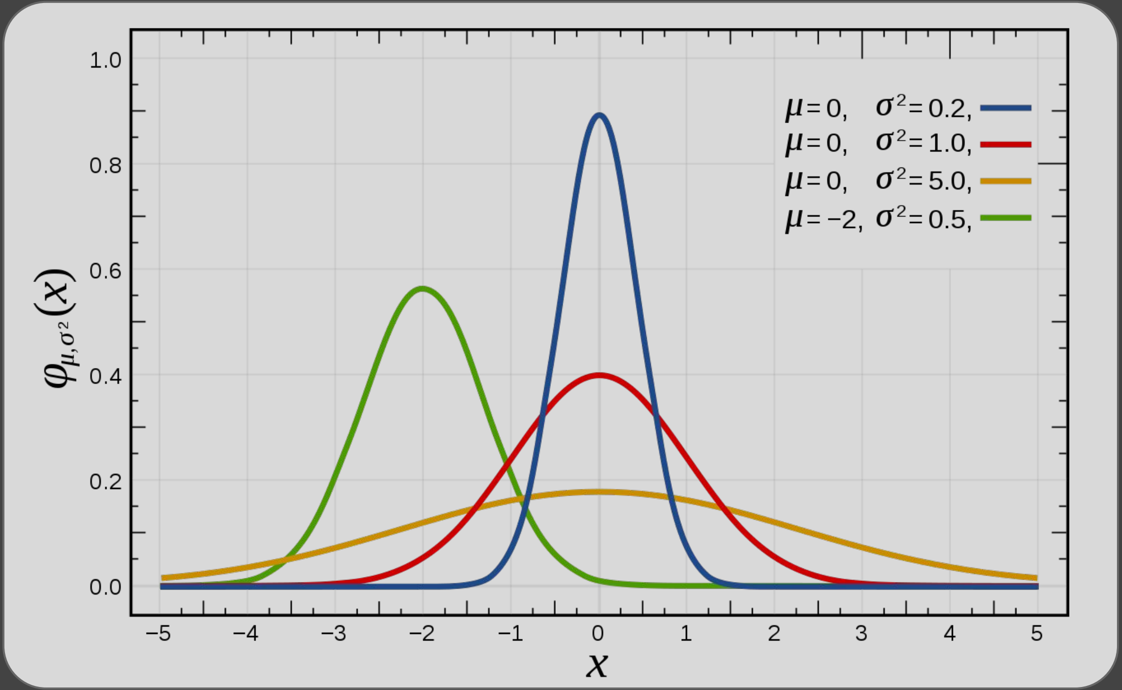
$P_N(n) \rightarrow$ PDF

$X \rightarrow Y$ $\quad$ CPD: $P(Y|X) = ?$

$n = Y - f(X)$ $\qquad P(Y|X) = P_N(n) = P_N(Y - f(X))$

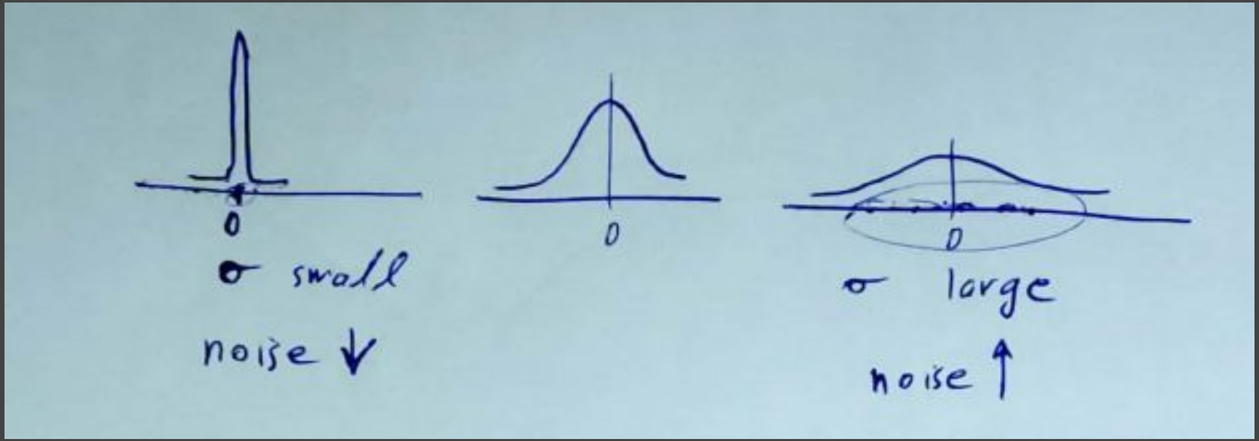
Gaussian Distribution

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



https://en.wikipedia.org/wiki/Normal_distribution

Gaussian Noise



Example 1: GPS

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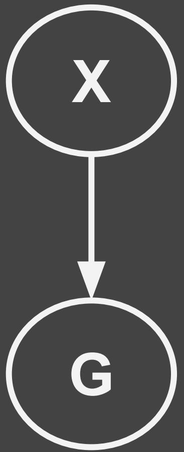

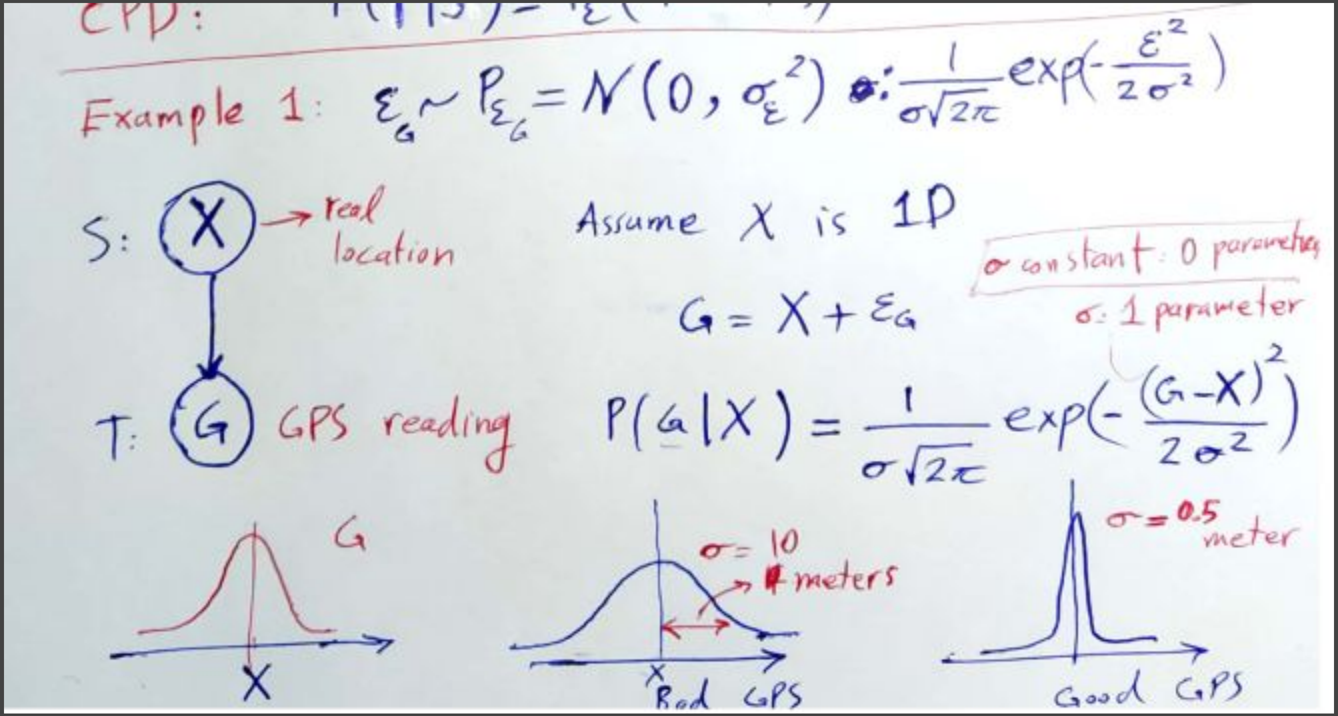

Example 1: GPS

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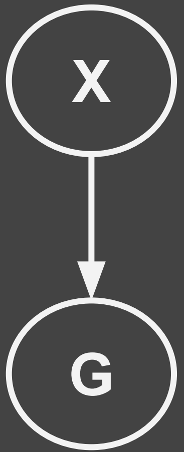
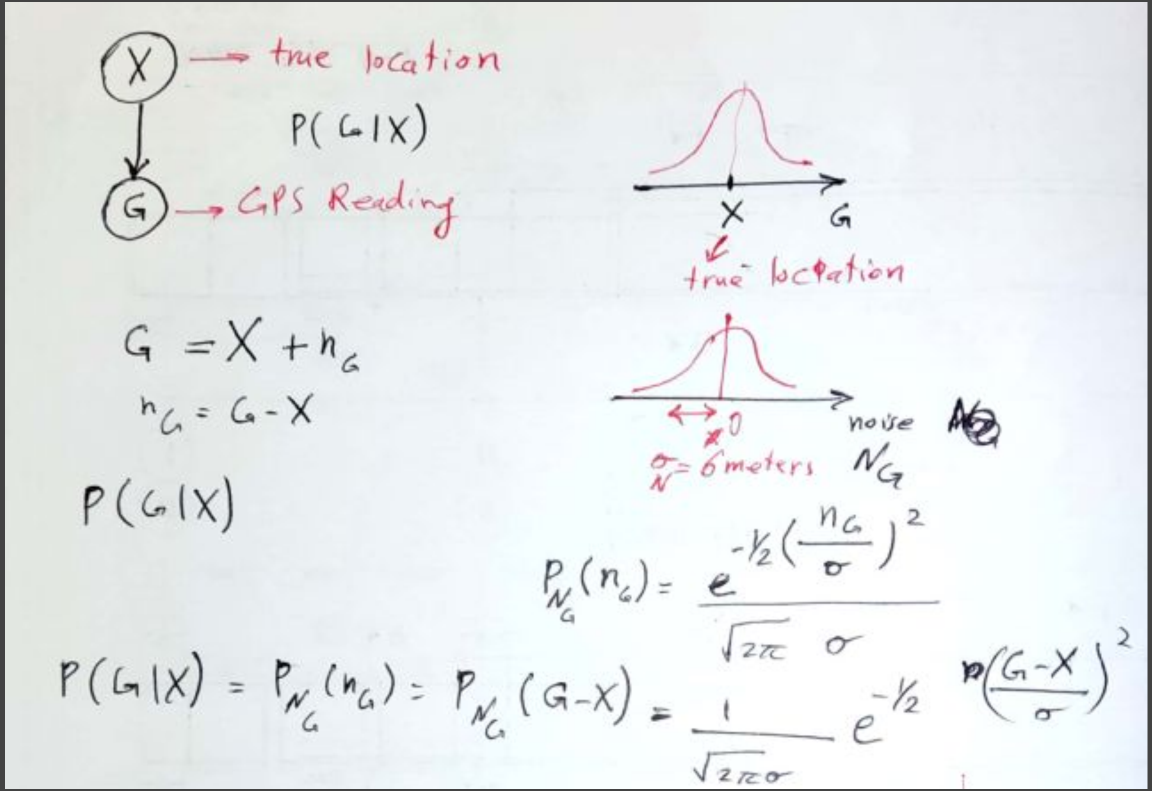
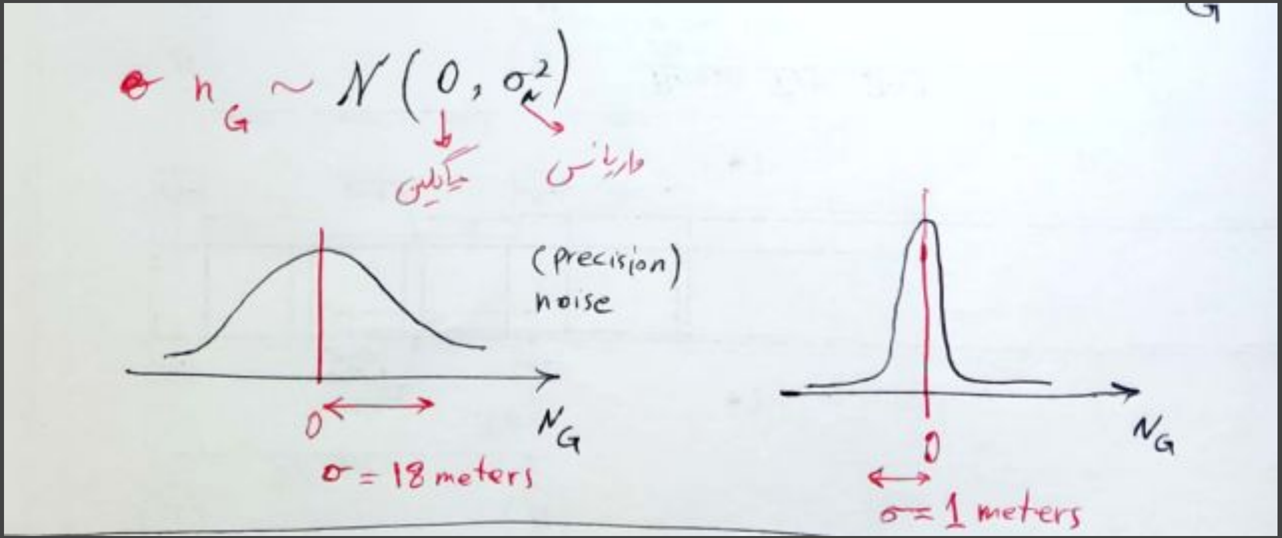
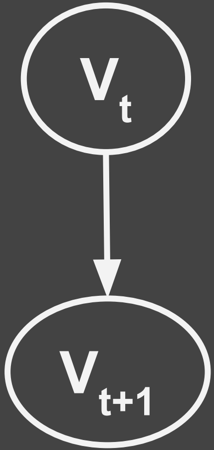


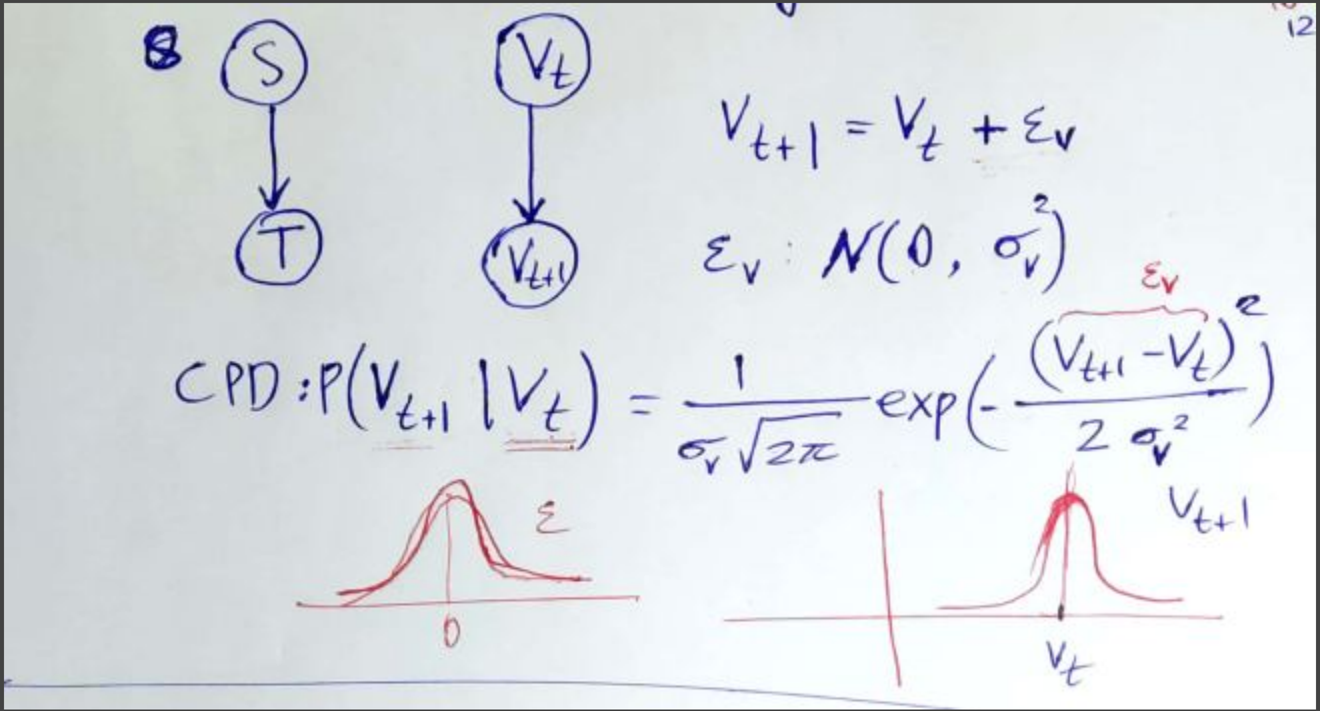


Example 1: GPS



Example 2: Constant Velocity





Example 3: Two variables

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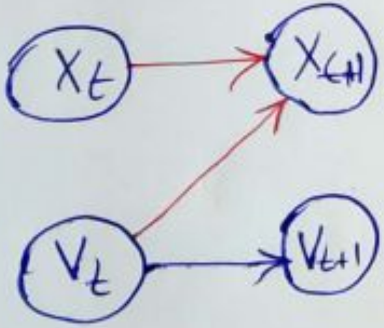
Eample 3:

$$V_{t+1} = V_t + \varepsilon_v$$

$$X_{t+1} = X_t + \Delta t\, V_t + \varepsilon_x$$



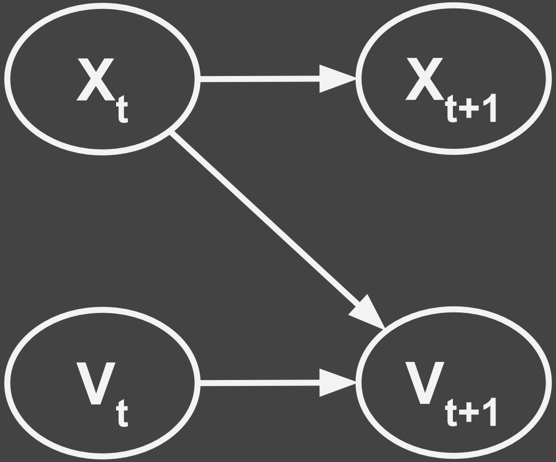
$$X_{t+1} \sim P(X_{t+1} \mid X_t, V_t) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(\frac{(X_{t+1} - X_t - \Delta t\, V_t)^2}{2\sigma_x^2}\right)$$

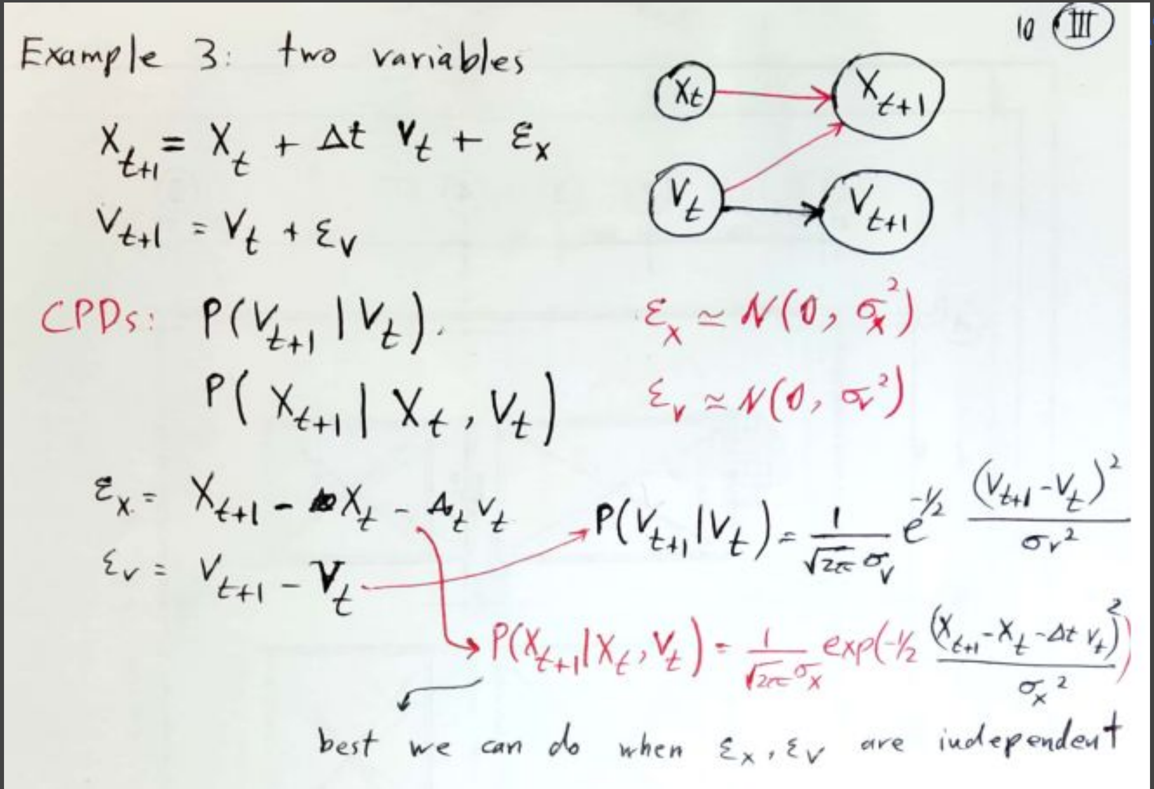
$$V_{t+1} \sim P(V_{t+1} \mid V_t) = \frac{1}{\sigma_v\sqrt{2\pi}} \exp\left(\frac{(V_{t+1} - V_t)^2}{2\sigma_v^2}\right)$$

Assuming $\varepsilon_v, \varepsilon_x$ are independent

$$P(\varepsilon_v, \varepsilon_x) = P(\varepsilon_v)\, P(\varepsilon_x)$$

Example 3: Two variables





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Example 3: two variables

$X_{t+1} = X_t + \Delta t \, V_t + \varepsilon_x$

$V_{t+1} = V_t + \varepsilon_V$

CPDs: $P(V_{t+1} | V_t)$. $\varepsilon_x \simeq N(0, \sigma_x^2)$

$P(X_{t+1} | X_t, V_t)$ $\varepsilon_v \simeq N(0, \sigma_v^2)$

$\varepsilon_x = X_{t+1} - X_t - \Delta_t V_t$

$\varepsilon_v = V_{t+1} - V_t$

$P(V_{t+1} | V_t) = \frac{1}{\sqrt{2\pi}\sigma_V} e^{-\frac{1}{2} \frac{(V_{t+1} - V_t)^2}{\sigma_V^2}}$

$P(X_{t+1} | X_t, V_t) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2} \frac{(X_{t+1} - X_t - \Delta t \, V_t)^2}{\sigma_X^2}\right)$

best we can do when $\varepsilon_x, \varepsilon_v$ are independent

Example 4: Dependent Errors

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$$X_{t+1} = X_t + \Delta t\, V_t + \varepsilon_x$$

$$V_{t+1} = V_t + \varepsilon_v$$

state at time $\underline{\underline{t}}$: $\begin{bmatrix} X_t \\ V_t \end{bmatrix}$

$S_t \in \mathbb{R}^2$

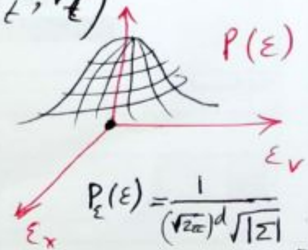
$$\begin{bmatrix} X_{t+1} \\ V_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_t \\ V_t \end{bmatrix} + \begin{bmatrix} \varepsilon_x \\ \varepsilon_v \end{bmatrix}$$

$$S_{t+1} = A\, S_t + \varepsilon$$

$S_{t+1}, S_t, \varepsilon \in \mathbb{R}^2$

$A \in \mathbb{R}^{2\times 2}$

$$P(S_{t+1} \mid S_t) = P(X_{t+1}, V_{t+1} \mid X_t, V_t)$$



$$\varepsilon = S_{t+1} - A\, S_t$$

$$\varepsilon \sim N(0, \Sigma) \longrightarrow \text{Covarica matrix}$$

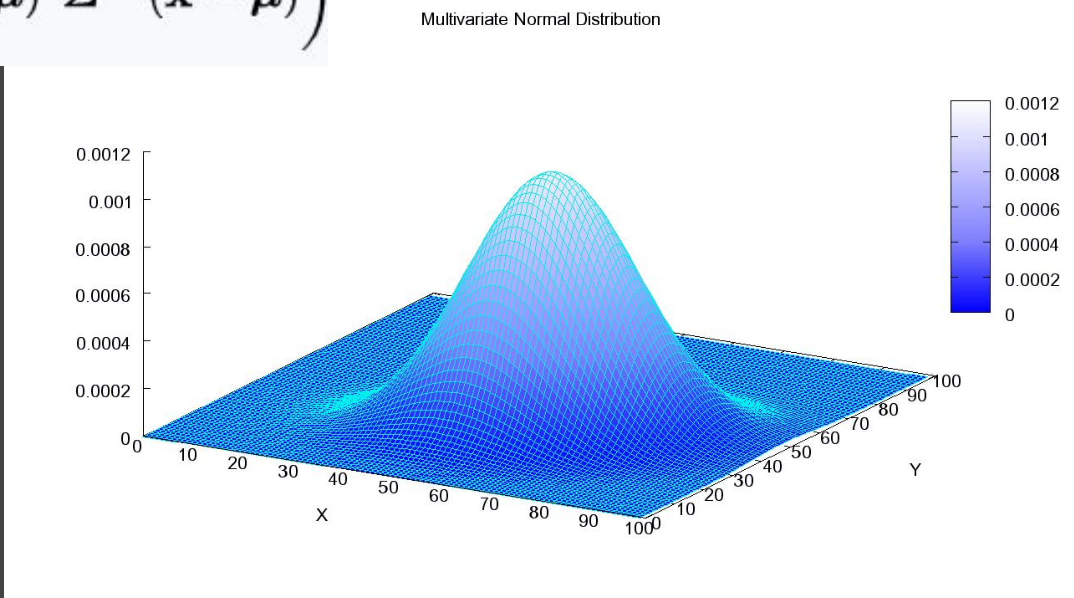
$$P_\varepsilon(\varepsilon) = \frac{1}{(\sqrt{2\pi})^d \sqrt{|\Sigma|}} \exp\left(-\tfrac{1}{2}\, \varepsilon^T \Sigma^{-1} \varepsilon\right)$$

CPD

$$P(S_{t+1} \mid S_t) = (\sqrt{2\pi})^{-d} |\Sigma|^{-1/2} \exp\left(-\tfrac{1}{2}(S_{t+1} - A S_t)^T \Sigma^{-1} (S_{t+1} - A S_t)\right)$$

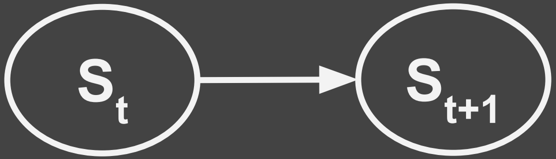
Multivariate Gaussian Distribution

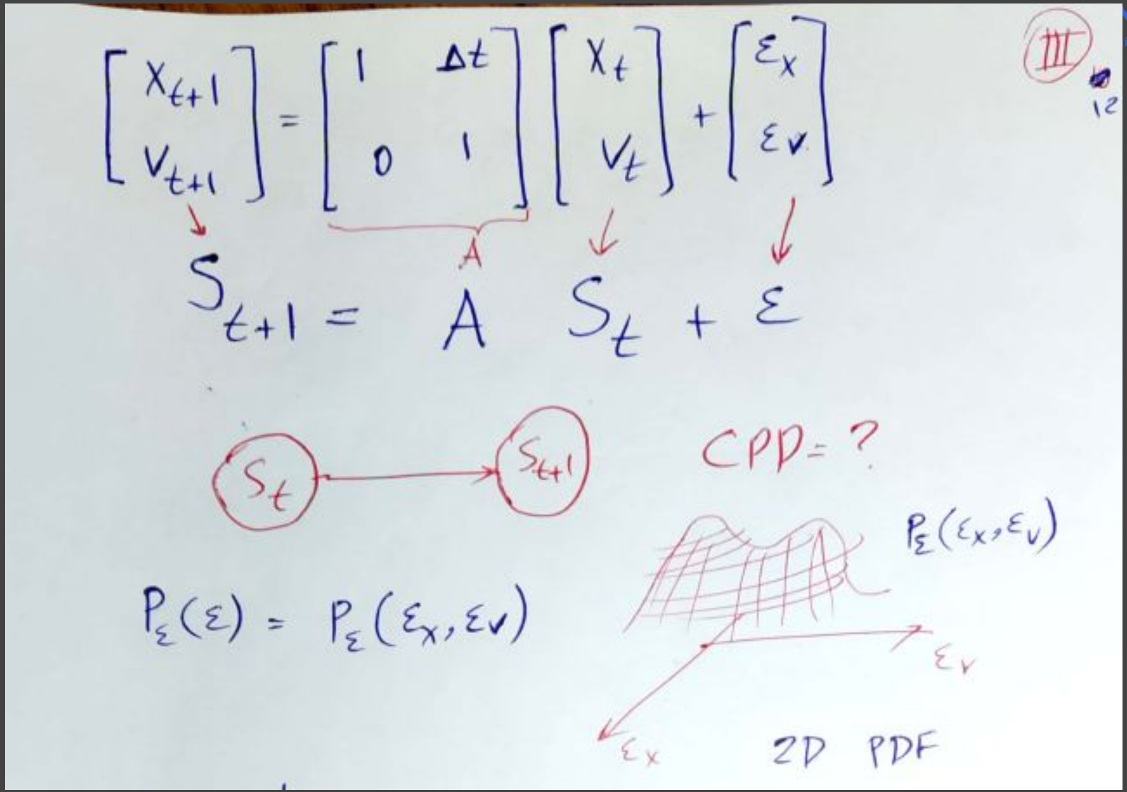
$$(2\pi)^{-k/2} \det(\boldsymbol{\Sigma})^{-1/2} \, \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$



https://en.wikipedia.org/wiki/Multivariate_normal_distribution

Example 4: Dependent Errors





Multivariate Gaussian Distribution

~~Assume~~ $P(X, Y)$

Simple Case: Assume X, Y are independent

$X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

$$P(X,Y) = P(X)P(Y) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{\frac{(X-\mu_x)^2}{-2\sigma_x^2}} \frac{1}{\sigma_y\sqrt{2\pi}} e^{\frac{(Y-\mu_y)^2}{-2\sigma_y}}$$

$$= \frac{1}{\sigma_x\sigma_y \underbrace{(\sqrt{2\pi})^2}_{2\pi}} \exp\left(\frac{(X-\mu_x)^2}{-2\sigma_x^2} + \frac{(Y-\mu_y)^2}{\sigma_y^2}\right)$$

covariance matrix

$$\Sigma = \begin{bmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{bmatrix}$$

X, Y are independent

$$\Sigma^{-1} = \begin{bmatrix}\sigma_x^{-2} & 0\\ 0 & \sigma_y^{-2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{\det(\Sigma)}\,(\sqrt{2\pi})^2} \exp\left(-\frac{1}{2}\left(\begin{bmatrix}X\\Y\end{bmatrix}-\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right)^T \Sigma^{-1}\left(\begin{bmatrix}X\\Y\end{bmatrix}-\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right)\right)$$

Multivariate Gaussian Distribution

$$= \frac{1}{(\det(\Sigma))^{1/2}\,(2\pi)^{2/2}} \exp\left(-\frac{1}{2}\,(S-\mu_S)^{T}\,\Sigma^{-1}\,(S-\mu_S)\right) \quad \text{IV}$$

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$$\underset{e\times 1}{S} = \begin{bmatrix} X \\ Y \end{bmatrix} \qquad \underset{1\times 2}{S^T} = \begin{bmatrix} X & Y \end{bmatrix}$$

$$\mu_S = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

Multivariate Gaussian Distribution

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$S \in \mathbb{R}^2$ X, Y are not independet

$\mu_S \in \mathbb{R}^2$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

$S \in \mathbb{R}^n$

$$= \frac{1}{\det(\Sigma)^{1/2} (2\pi)^{n/2}} \exp\left(-\frac{1}{2}(S-\mu)^T \Sigma^{-1} (S-\mu)\right)$$

$\mu \in \mathbb{R}^n$

$\Sigma \in \mathbb{R}^{n \times n}$

Dependent Errors

$P(\varepsilon_x, \varepsilon_v)$ $\varepsilon_x, \varepsilon_v$ are not independet

$$S_{t+1} = A S_t + \varepsilon \qquad \varepsilon \sim N(0, \Sigma)$$

$$\mu = 0$$

$$\begin{bmatrix} x_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ v_t \end{bmatrix} + \begin{bmatrix} \varepsilon_x \\ \varepsilon_v \end{bmatrix}$$

$$\underset{P(S_{t+1} \mid S_t)}{CPD} = \frac{1}{\det(\Sigma)^{1/2} (2\pi)^{n/2}} \exp\left(-\frac{1}{2} (S_{t+1} - A S_t)^T \Sigma^{-1} (S_{t+1} - A S_t)\right)$$

Observation models

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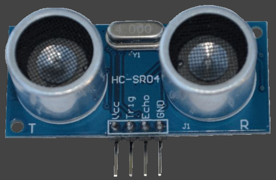
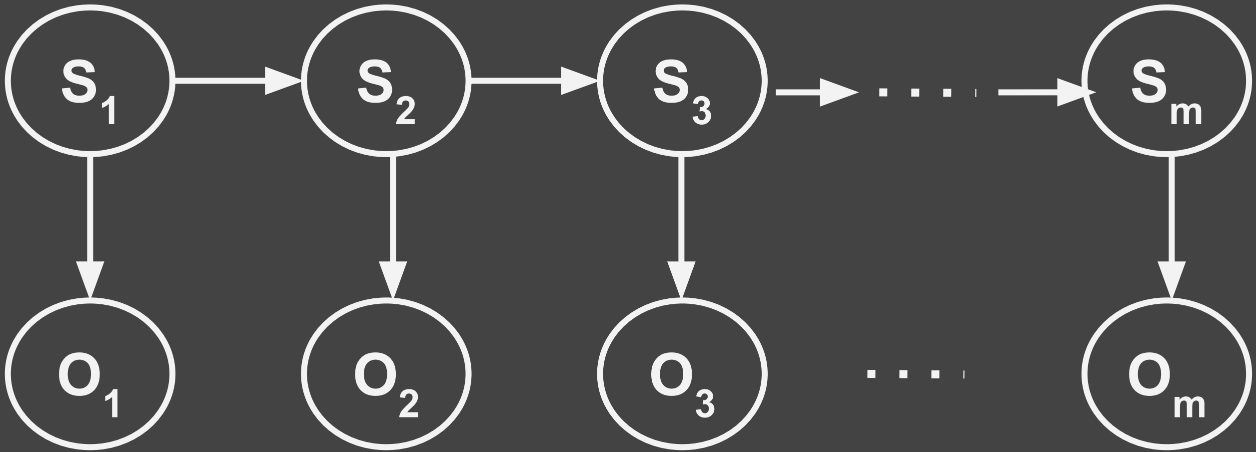
Distance

Ultrasonic sensor Measurement

LiDAR sensor Measurement



Sequential State-Observation Model



$P(S_{i+1} \mid S_i)$: Transition Model

$P(O_{i+1} \mid O_i)$: Observation Model